JOURNAL OF APPROXIMATION THEORY 13, 421–430 (1975)

# Some Nonlinear Spline Approximation Problems Related to N-Widths

CHARLES K. CHUI AND PHILIP W. SMITH

*Department of Mathematics, Texas A & M University, College Station, Texas 77843*

*Communicated by Oved Shisha*

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

## 1. INTRODUCTION

Let $X$ be a normed linear space with norm $\|\ \|$. For a positive integer $N$, the $N$-width of a set $B$ in $X$ is defined by

$$d_N(B) = \inf_{M_n} \sup_{x \in B} \inf_{y \in M_N} \| x - y \| , \tag{1.1}$$

where the infimum is taken over all $N$-dimensional affine varieties $M_N$ in $X$ (cf. [8]). The concept of $N$-width was introduced by Kolmogorov [7]; its idea may be viewed as that of finding an extremal $N$-dimensional subspace of $X$ which globally approximates the set $B$. For a Hilbert space $X$, Kolmogorov [7] and Jerome [5, 6] studied the $N$-widths of ellipsoids defined by differential operators.

Let $L$ be the differential operator

$$L = \prod_{j=1}^{r} (D - b_j), \tag{1.2}$$

with real constants $b_1, ..., b_r$, where $D = d/(dt)$; and let

$$B(L) = \{x \in W^{r,\infty}[-1, 1]: \| Lx \|_\infty \leqslant 1\} \tag{1.5}$$

where, as usual, $W^{r,\infty}[-1, 1]$ denotes the Sobolev space of functions which are $r$-fold integrals of $L^\infty[-1, 1]$ functions. In this paper we will derive exact expressions for the $N$-widths of the sets $B(L)$ in $X = C[-1, 1]$, as well as exhibit an extremal subspace.

Solutions to the width problem are a consequence of the following nonlinear approximation problem, which is independently of some interest. Let $S(n, L)$ be a set of exponential splines relative to the differential operator $L$

defined explicitly in (2.1) of the next section, and let $V$ be some function satisfying $(DL)\, v(t) = \delta(t - 1)$. Our approximation problem is

$$\inf_{x \in S(n,L)} \| v - x \| = \gamma(n, L). \tag{1.4}$$

The main result of this paper is that $d_N(B(L)) = \gamma(N - r, L)$ when $\prod_{i=1}^{r} b_i = 0$. In [10], Tihomirov considered the special case $L = D^r$, with a more restrictive nonlinear approximation problem than (1.4).

## 2. The Minimization Problem

In this section we pose and solve a nonlinear best approximation problem in $C[-1, 1]$. The solution to this problem will be instrumental in solving the width problem as mentioned in the introduction. For $r = 1, 2, \ldots$, set

$$x(t) = \sum_{i=0}^{r-1} \beta_i t^i + \sum_{i=1}^{m} \alpha_i (t - t_i)_+^r, \qquad -1 \leqslant t_1 < t_2 < \cdots < t_m \leqslant 1, \tag{2.1}$$

and let $S(n, r)$ be the collection of all splines $x$ of the form (2.1) with $m \leqslant n$ and $\alpha_i = \pm(2/r!)$. The $t_i$'s will be called the knots of the spline function $x$. The best approximation problem is then

$$\inf_{x \in S(n,r)} \|((t + 1)^r/r!) - x(t)\| . \tag{2.2}$$

The solutions to this problem can be viewed as generalizations of the Chebyschev polynomials. In [10], Tihomirov considered a similar problem with the important exception that the $\alpha_i$'s were required to alternate in sign. However, as we will show, this alternation is built into the solution to (2.2). We will establish the following

Theorem 2.1. *There is a solution $x_*$ to* (2.2) *which has exactly $n$ knots and the curve of the error function $x_{nr}(t) = ((t + 1)^r/r!) - x_*(t)$ has $n + r + 1$ alternation. Furthermore, the $\alpha_i$'s alternate in sign and $\| D^r x_{nr} \|_\infty = 1$.*

We will give the entire proof of this theorem so that the method can be used for generalization to a larger class of operators in the next section. To prove Theorem 2.1 we will first establish five lemmas.

Lemma 2.1. *The problem* (2.2) *has a solution in $S(n, r)$.*

The proof of this lemma follows from a standard compactness argument. The lemma to follow guarantees that there is a solution $x_*$ with exactly $n$ knots in $(-1, 1)$.

LEMMA 2.2. *There is a solution $x_*$ in $S(n, r)$ to problem* (2.2) *which has precisely $n$ distinct knots in the open interval* $(-1, 1)$.

*Proof.* Let $x_*$ solve (2.2). Without loss of generality, we may assume that all the $m$ $(m \leqslant n)$ knots of $x_*$ lie in $(-1, 1)$. Let $y(t) = ((t+1)^r/r!) - x_*(t)$. If $m < n$, then we may add to $x_*$ the sum $\sum_{j=m+1}^{n} \alpha_j(t - t_j)_+^r$, where for all $j$, $\alpha_j = (2/r!) \operatorname{sgn} y(1)$ if $y(1) \neq 0$ and $\alpha_j = 2/r!$ if $y(1) = 0$, and the $t_j$'s, $t_{m+1} < \cdots < t_n < 1$, are so close to 1 that the value $\| y \|$ is not increased.

We now derive a fundamental approximation theoretic result which links $x_{nr}(t) = ((t+1)^r/r!) - x_*(t)$ to an $(n+r)$-dimensional spline subspace of $C[-1, 1]$. Let $M$ be the subspace generated by $\{1, t, \ldots, t^{r-1}, (t - t_1)_+^{r-1}, \ldots, (t - t_n)_+^{r-1}\}$ where $\{t_1, \ldots, t_n\}$ are the knots of a solution $x_*$ to (2.2) as given by Lemma 2.2. We may now state

LEMMA 2.3. *Let $x_{nr}$, $r \geqslant 2$, be the error $((t+1)^r/r!) - x_*(t)$ where $x_*$ is a solution of* (2.2) *with $n$ distinct knots $t_1, \ldots, t_n$. Then the zero function $\theta$ is a best approximant to $x_{nr}$ from $M$.*

*Proof.* Following the ideas in [10] we let $G$: $R^{n+r} \to C[-1, 1]$ be defined by

$$G(\gamma) = -\sum_{i=0}^{r-1} \gamma_i t^i - \sum_{i=1}^{n} \alpha_i(t - \gamma_{i+r-1})_+^r + \frac{(t+1)^r}{r!} \tag{2.3}$$

where $\gamma = (\gamma_0, \ldots, \gamma_{n+r-1})$. We suppose that the $\alpha_i$'s are chosen so that $G(\gamma^*) = x_{nr}$. It is easy to see that $G$ is Fréchet differentiable at $\gamma^*$ and further that

$$G'(\gamma^*)(\eta) = -\sum_{i=0}^{r-1} \eta_i t^i + r \sum_{i=1}^{n} \alpha_i \eta_{i+r-1}(t - \gamma_{i+r-1}^*)_+^{r-1}, \tag{2.4}$$

where $\eta = (\eta_0, \ldots, \eta_{r+n-1})$. Since $\| G(\gamma)\|$ has a global minimum at $\gamma = \gamma^*$ we conclude that

$$\| G(\gamma^*) + G'(\gamma^*)(\eta)\| \geqslant \| G(\gamma^*)\|, \tag{2.5}$$

for all $\eta \in R^{n+r}$. Since $G'(\gamma^*)(R^{n+r}) = M$, we conclude that $\theta$ is a best approximation to $x_{nr} \equiv G(\gamma^*)$ from $M$.

The next lemma is a standard result in approximation theory. For example see Singer [9].

LEMMA 2.4. *There exist $p$ $(p \leqslant n + r + 1)$ points $\tau_1, \ldots, \tau_p$ with $-1 \leqslant \tau_1 < \cdots < \tau_p \leqslant 1$, $p$ nonzero real numbers $\mu_1, \ldots, \mu_p$, and a functional $F$ of the form*

$$F = \sum_{i=1}^{p} \mu_i \delta(t - \tau_i) \tag{2.6}$$

*in* $C^*[-1, 1]$, *satisfying* (a) $\|F\| = 1$, (b) $F \in M^\perp$, *and* (c) $F(x_{nr}) = \|x_{nr}\|$. *Here,* $\delta(t - \tau_i)$ *represents the purely atomic measure with weight one at* $\tau_i$.

Next we consider the function $f$ defined by

$$f(t) = \frac{1}{(r-1)!} \sum_{i=1}^{p} \mu_i (t - \tau_i)_+^{r-1}, \tag{2.7}$$

where $p$, the $\mu_i$'s and the $\tau_i$'s are as in Lemma 2.4. Clearly, $f$ vanishes on $(-\infty, -1]$ and $f$ is in $C^{r-2}(R^1)$. It is easy to see (cf. [10]) that $f$ also vanishes on $[1, \infty)$. Indeed, for $t \geqslant 1$, we may remove the plus subscript in (2.7), expand the binomial terms, and collect the result in powers of $t$, obtaining

$$f(t) = \frac{(-1)^{r-1}}{(r-1)!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-t)^j \sum_{i=1}^{p} \mu_i \tau_i^{r-1-j}, \tag{2.8}$$

and observe that since $F$ annihilates polynomials of degree no greater than $r - 1$, the last sum in (2.8) is zero.

We now state the final lemma.

LEMMA 2.5. *The number of knots of the spline function* $f$ *in* (2.7) *is* $p = n + r + 1$.

The above lemma was proved in [10]. However, we feel that the exposition there was somewhat unsatisfactory and for that reason we include a proof. Let $I = [a, b]$ be the smallest nontrivial interval to the left of 1 so that

$$f^{(j)}(a-) = f^{(j)}(b+) = 0, \qquad j = 0, 1,..., r - 1. \tag{2.9}$$

We wish to show that $I = [-1, 1]$. Suppose this is not the case. Then noting that

$$D^r f = F, \tag{2.10}$$

we have, for any $y \in M$, by Lemma 2.4,

$$0 = F(y) = \int_{-1}^{1} D^r f(t)\, y(t)\, dt = (-1)^r \int_{-1}^{1} f(t)\, D^r y(t)\, dt. \tag{2.11}$$

Here, if necessary, the integral can be taken in a natural way over $[-1 - \epsilon, 1 + \epsilon]$ for small positive $\epsilon$. In particular, taking $y(t) = (t - t_i)_+^{r-1}$ we obtain $f(t_i) = 0$, for $i = 1,..., n$. Suppose there are exactly $q$ $t_i$'s in $(a, b)$. Then $f$ has at least $q$ isolated zeros there. Using Rolle's theorem $r - 2$ times and recalling that $f$ as well as the first $r - 2$ derivatives of $f$ vanish at $a$ and $b$, we conclude that $f^{(r-2)}$ has at least $q + r - 2$ isolated zeros in $(a, b)$. Hence, for all small $\delta > 0$, $f$ has at least $q + r + 1$ knots in

$[a - \delta, b + \delta]$. Since at each knot of $f$, $|x_{nr}|$ attains a global maximum from (a) and (c) of Lemma 2.4, we know that $x_{nr}$ has at least $q + r$ maxima and/or minima in the interior of $[-1, 1]$. Of course, if neither $a = -1$ nor $b = 1$ then $x_{nr}$ would have $q + r + 1$ maxima and/or minima in the interior of $[-1, 1]$. We will assume that $a > -1$ and $b = 1$. The cases where $-1 < a < b < 1$ and $-1 = a < b < 1$ can be treated similarly. It follows that $x_{nr}^{(1)}$ must have at least $q + r$ zeros in $[a - \delta, b)$ for all small $\delta > 0$. In fact, from the definition of $S(n, r)$ it can easily be seen that $|x_{nr}^{(r)}(t)| \geqslant 1$ and hence $x_{nr}^{(1)}$ must change sign at the $q + r$ interior maxima and/or minima in $[a - \delta, b)$. Using Rolle's theorem $r - 1$ times we can conclude that $x_{nr}^{(r-1)}$ has at least $q + 2$ sign changes in $[a - \delta, b)$ for any $\delta > 0$. In particular, since $x_{nr}^{(r-1)}$ is piecewise linear we can conclude that $x_{nr}$ must have at least $q + 1$ knots in $(a, b)$. But this is contrary to our assumption that there are exactly $q$ knots of $x_{nr}$ in $(a, b)$. It must therefore be the case that $a = -1$ and $b = 1$. Now by exactly the same argument as above we conclude that $f$ has $n$ isolated zeros in $(-1, 1)$ and thus $f$ has $n + r + 1$ knots in $[-1, 1]$.

We are now in a position to prove Theorem 2.1. Lemma 2.1 and Lemma 2.2 guarantee that there is a solution $x_*$ to (2.2) with exactly $n$ knots. From the proof of Lemma 2.5 it is easy to see that $x_{nr}$ alternates sign at least $n + r + 1$ times. Since $x_{nr}$ has only $n$ knots, it can alternate no more than $n + r + 1$ times. The only way $x_{nr}$ can alternate $n + r + 1$ times is for the $\alpha_i$'s to alternate in sign with $\alpha_i = (-1)^{i+1}(2/r!)$ for $i = 1,..., n$. Since the $\alpha_i$'s have this structure it is easy to see that $\| D^r x_{nr} \|_\infty = 1$.

## 3. The Generalized Minimization Problem

In this section, problem (2.2) is reformulated in a more general setting with the differential operator $D^r$ replaced by

$$L = \prod_{j=1}^{r} (D - b_j), \tag{3.1}$$

where $b_1,..., b_r$ are real numbers. Throughout this section, we will assume that $b_1 = 0$. When all the $b_i$'s are nonzero, the first four lemmas in this section are still valid. In Section 4, we will indicate how the case when all the $b_i$'s are nonzero can be treated.

Consider the set of functions $x$ of the form

$$x(t) = \sum_{i=1}^{m} \alpha_i v(t - t_i) + \sum_{i=1}^{r} \beta_i w_i(t), \tag{3.2}$$

with $-1 \leqslant t_1 < \cdots < t_m \leqslant 1$, where $\{w_1,..., w_r\}$ is a basis of the nullspace of the operator $L$ and where $v \in W^{r,\infty}[-1, 1]$, $v(t) = 0$ for $t < 0$, and

$$Lv(t) = (t)_+^0 . \tag{3.3}$$

We let $S(n, L)$ denote the set of functions $x$ of the form (3.2) with $m \leqslant n$ and $\alpha_i = \pm 2$. The best approximation problem then becomes

$$\inf_{x \in S(n,L)} \| v(t+1) - x(t) \| . \tag{3.4}$$

The theorem corresponding to Theorem 2.1 is

THEOREM 3.1. *There is a solution $x_*$ to* (3.4) *which has exactly $n$ knots and the curve of the error function $x_n(L)(t) \equiv v(t+1) - x_*(t)$ has $n + r + 1$ alternation. Furthermore, the $\alpha_i$'s alternate in sign and $\| Lx_n(L)\|_\infty = 1$.*

The proof of this theorem follows quite closely the proof of Theorem 2.1. We note that operators of the form (3.1) satisfy a generalized Rolle's theorem as indicated in the following proposition which is proved in [4].

PROPOSITION. *Let $y \in C^r[-1, 1]$ with $j \geqslant r$ sign changes, then $Ly$ has $j - r$ sign changes.*

Since Rolle's Theorem was used as a major tool in proving Theorem 2.1, it is easy to see the corresponding uses of the above proposition. We list the lemmas which are necessary for the proof of Theorem 3.1 and comment on the modifications needed to adapt the proofs of the corresponding lemmas in Section 2.

LEMMA 3.1. *The problem* (3.4) *has a solution in $S(n, L)$.*

LEMMA 3.2. *There is a solution $x_*$ in $S(n, L)$ to problem* (3.4) *which has precisely $n$ distinct knots in* $(-1, 1)$.

The proof of this lemma just relies on the fact that the $\alpha_i$'s may be chosen to be either $+1$ or $-1$ and hence we may place knots very close to 1 without increasing the norm of the error.

Let $M = M(L, n)$ be the subspace generated by $\{w_1,..., w_r, v^{(1)}(t - t_1),..., v^{(1)}(t - t_n)\}$ where $\{w_1,...,w_r\}$ is a basis for the null-space of the operator $L$ and $t_1,..., t_n$ are the knots of a solution $x_*$ to problem (3.4). Then we have the following approximation result.

LEMMA 3.3. *Let $x_n(L)$ be the error $v(t+1) - x_*(t)$ where $x_*$ is a solution of problem* (3.4) *with $n$ distinct knots $t_1,..., t_n$. Then $\theta$ is a best approximant to $x_n(L)$ from $M$.*

This lemma follows from examining the Fréchet derivative of the function $G$, $G: R^{n+r} \to C[-1, 1]$, defined by

$$G(\gamma) = v(t) - \sum_{i=1}^{r} \gamma_i w_i - \sum_{i=1}^{n} \alpha_i v(t - \gamma_{i+r}),$$

where $\gamma = (\gamma_1, ..., \gamma_{n+r})$. It is easy to see that if $G(\gamma_*) = x_n(L)$ then $G'(\gamma_*)(R^{n+r}) = M$.

LEMMA 3.4. *There are $p \leqslant n + r + 1$ points $-1 \leqslant \tau_1 < \tau_2 < \cdots < \tau_p \leqslant 1$, $p$ nonzero numbers $\mu_i$, and a functional $F$ of the form*

$$F = \sum_{i=1}^{p} \mu_i \delta(t - \tau_i) \tag{3.5}$$

*in $C^*[-1, 1]$, satisfying*

(a) $\| F \| = 1$,

(b) $F \in M^{\perp}$, *and*

(c) $F(x_n(L)) = \| x_n(L) \|$.

We now let $L^*$ be the formal adjoint of $L$, and let $v^* \in W^{r-1,\infty}[-3, 3]$ with $v^*(t) \equiv 0$ for $t \leqslant 0$ satisfying

$$L^* v^*(t) = \delta(t). \tag{3.6}$$

Then we define

$$f(t) = \sum_{i=1}^{p} \mu_i v^*(t - \tau_i). \tag{3.7}$$

Clearly, $L^* f = F$ and $f(t) = 0$ for $t \in (-2, -1]$. If the numbers $b_1, ..., b_r$ are distinct then

$$v^*(t) = \sum_{j=1}^{r} c_j e^{-b_j t} \tag{3.8}$$

and hence for $t \geqslant 1$

$$f(t) = \sum_{i=1}^{p} \mu_i \sum_{j=1}^{r} c_j e^{-b_j(t-\tau_i)} = \sum_{j=1}^{r} c_j e^{-b_j t} \sum_{i=1}^{p} \mu_i e^{b_j \tau_i}. \tag{3.9}$$

The last sum in (3.9) is identically zero since $e^{b_j t}$ is in the nullspace of $L$. In general, when the $b_i$'s are not necessarily distinct, we can also conclude as above, using the binomial expansion and regrouping, that $f(t) = 0$ for $t \in [1, 2)$.

We may now state the final lemma that we need for the proof of Theorem 3.1.

LEMMA 3.5. *The number of knots of the exponential spline function $f$ in* (3.7) *is* $p = n + r + 1$.

The proof of this lemma proceeds similarly to the proof of Lemma 2.5. For instance, as in (2.11), we have for $y \in M$

$$0 = F(y) = \int_{-1}^{1} L^*f(t)\, y(t)\, dt = \pm \int_{-1}^{1} f(t)\, Ly(t)\, dt. \tag{3.10}$$

Choosing $y(t) = v^{(1)}(t - t_i)$, then $Ly(t) = \delta(t - t_i)$ and hence, $f(t_i) = 0$ for $i = 1,..., n$. Now, by using the generalized Rolle's theorem, Lemma 3.5 follows. Also, Theorem 3.1 now follows by arguments similar to those used in proving Theorem 2.1.

We now consider the case when all the $b_i$'s are nonzero. The only problem we encounter is in the proof of Lemma 3.5. In that lemma, we note that $| Lx_n(L)(t)| \geqslant 1$ implies that $x_n(L)$ cannot be constant on a nontrivial subinterval. This is no longer the case when $\prod_{j=1}^{r} b_j \neq 0$. However, at that stage of the proof we are only interested in whether $| x_n(L)|$ is constantly equal to $\| x_n(L)\|$, on a subinterval $J$, say. If this is the case, we have

$$| L(x_n(L))(t)| = \left| \prod_{j=1}^{r} b_j \right| \| x_n(L)\|, \tag{4.3}$$

for $t \in J$. But $\| x_n(L)\| \to 0$ as $n \to \infty$. Hence, there is an integer $n_0$ such that for all $n \geqslant n_0$, $| \prod_{j=1}^{r} b_j | \; \| x_n(L)\|$ is less than one. This means that for $n \geqslant n_0$, $x_n(L)$ cannot be constant when $| x_n(L)|$ attains its maximum. Thus, for $n \geqslant n_0$ Lemma 3.5 is also valid for $\prod_{j=1}^{r} b_j \neq 0$.

## 4. $N$-WIDTHS OF $B(L)$

Let $L = \prod_{j=1}^{r} (D - b_j)$, where $b_1 ,..., b_r$ are real numbers and $b_1 = 0$. The case when all the $b_i$'s are nonzero will be discussed at the end of this section. Let $B(L)$ be as in (1.2), and let $M(L, n)$ be the subspace of $C[-1, 1]$ and $x_{N-r}(L)$ be the error in the solution of problem (3.4) as defined in Section 3. We have the following result on the $N$-widths of $B(L)$.

THEOREM 4.1. *For* $N \leqslant r - 1$, $d_N(B(L)) = \infty$, *and for* $N \geqslant r$, $d_N(B(L)) = \| x_{N-r}(L)\|$.

This theorem follows directly by applying the generalized Rolle's theorem in the proof of Theorem 2 in [10]. In the course of the proof it can be seen that the subspace $M(L, N - r)$, for $N \geqslant r$, is an *extremal approximating* $N$-dimensional *subspace* of $C[-1, 1]$ in the sense of $N$-width (cf. [8]); that is, we have the following corollary.

COROLLARY. *For* $N \geqslant r$,

$$d_N(B(L)) = \sup_{x \in B(L)} \inf_{y \in M(L, N-r)} \| x - y \| . \tag{4.1}$$

We wish to point out that $M(L, N - r)$ is an exponential spline subspace, and further that there is a linear projection $P$ mapping $C[-1, 1]$ onto $M(L, N - r)$ so that

$$\sup_{x \in B(L)} \| x - Px \| = d_N(B(L)). \tag{4.2}$$

In fact, the projection $P$ is defined by interpolation at the $N$ zeros of $X_{N-r}(L)$.

When $\prod_{j=1}^{r} b_j \neq 0$, we have the following theorem.

THEOREM 4.2. *Let*

$$L = \prod_{j=1}^{r} (D - b_j),$$

*where* $b_1, \ldots, b_r$ *are arbitrary real numbers. Then there is a positive integer* $n_0$ *such that* $d_N(B(L)) = \| x_{N-r}(L) \|$ *for all* $N \geqslant n_0$.

## 5. FINAL REMARKS

There are many interesting, and perhaps quite important, questions yet to be answered concerning these problems. The first natural question is whether the results in this paper are still true when the operator $L$ is $p(D)$ where $p$ is a polynomial with real constant coefficients and nonreal roots. If the coefficients of the polynomial $p$ are functions, the minimization and width problems for $L = p(D)$ seem to be quite complicated but important. More generally, for which linear operators $L$ will Theorem 4.1 remain valid? From a solution $x_*$ to problem (3.4), we can construct the extremal subspace $M(L, n)$. However, we cannot compute $x_*$ in a closed form. It would be of interest to know precisely the location of the knots of $x_*$ and to know whether $x_*$ is unique or not. It is also of interest to know the precise rate of decrease of $d_N(B(L))$ as $N \to \infty$. Since there has been much recent interest in nonlinear approximation, the exact (or exact asymptotic) distances in

(1.4) can possibly be calculated yielding the rate of decrease of the $N$-widths. In this area, Braess [1, 2, 3] has studied nonlinear approximation problems with restrictions on the coefficients.

## REFERENCES


1. D. BRAESS, Chebyshev approximation by $\gamma$-polynomials, *J. Approximation Theory*, **9** (1973), 20–43.
2. D. BRAESS, Chebyshev approximation by $\gamma$-polynomials, II, *J. Approximation Theory*, **11** (1974), 16–37.
3. D. BRAESS, Kritische Punkte bei der nichtlinearen Tschebyscheff-Approximation, *Math. Z. Bd.* **132** (1973), 327–341.
4. R. B. HOLMES, "A Course in Optimization and Best Approximation," Springer-Verlag, New York, 1972.
5. J. JEROME, On the $L_2$ $n$-width of certain classes of functions of several variables, Ph.D. Dissertation, Purdue University, Purdue, IA, 1966.
6. J. JEROME, Asymptotic estimates of the $n$-widths in Hilbert spaces, *Proc. Amer. Math. Soc.*, **33** (1972), 367–372.
7. A. N. KOLMOGOROV, Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse, *Ann. of Math.* **37** (1936), 107–111.
8. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1966.
9. I. SINGER, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. Springer-Verlag, New York, 1970.
10. V. M. TIHOMIROV, Best methods of approximation and interpolation of differentiable functions in the space $C[-1, 1]$, *Math. USSR Sbornik*, **9** (1969), 275–289.